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Noncommutative superspace, $\mathcal{N} = 1/2$ supersymmetry, field theory and string theory

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Abstract: We deform the standard four dimensional $\mathcal{N} = 1$ superspace by making the odd coordinates θ not anticommuting, but satisfying a Clifford algebra. Consistency determines the other commutation relations of the coordinates. In particular, the ordinary spacetime coordinates x cannot commute. We study chiral superfields and vector superfields and their interactions. As in ordinary noncommutative field theory, a change of variables allows us to express the gauge interactions in terms of component fields which are subject to standard gauge transformation laws. Unlike ordinary noncommutative field theories, the change of the lagrangian is a polynomial in the deformation parameter. Despite the deformation, the noncommutative theories still have an antichiral ring with all its usual properties. We show how these theories with precisely this deformation arise in string theory in a graviphoton background.

Keywords: D-branes, Space-Time Symmetries, Superspaces.

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1. Introduction

One of the objectives of string theory is to understand the nature of spacetime. It is therefore natural to ask how to add structure to standard $\mathbb{R}^4$, and how this space can be deformed. An example of an added structure is the addition of anticommuting spinor coordinates θ^α and $\overline{\theta}^{\dot{\alpha}}$ (we follow the notation of [1]). The resulting space is known as superspace. An example of a deformation is to make $\mathbb{R}^4$ noncommutative (for a review, see e.g. [2]). The combination of these two ideas; i.e. that the anticommuting coordinates θ form a Clifford algebra

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta} \tag{1.1}$$

has been explored by many people (for a partial list, see [3]–[16]). Here we will examine the consequences of this deformation. We will see that because of (1.1) the $\mathbb{R}^4$ coordinates x^μ cannot commute. Instead, the chiral coordinates

$$y^\mu = x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot{\alpha}} \overline{\theta}^{\dot{\alpha}} \tag{1.2}$$

can be taken to commute. Our noncommutative superspace is defined as follows. All the (anti)commutators of y^μ, θ^α and $\overline{\theta}^{\dot{\alpha}}$ vanish except (1.1). We will refer to the space with nonzero C as noncommutative and with zero C as commutative (although in fact, it is partially commutative and partially anticommutative).

In section 2 we will motivate the definition of the noncommutative superspace and will explore some of its properties. In particular, we will show that the deformation (1.1) breaks half the supersymmetry. Only the Q_α supercharges are conserved, while the $\overline{Q}_{\dot{\alpha}}$ supercharges are broken. Since half of $\mathcal{N} = 1$ supersymmetry is broken, we can refer to the unbroken Q supersymmetry as $\mathcal{N} = 1/2$ supersymmetry. We will also mention some generalizations of the minimal deformation. In section 3 we will study chiral and antichiral superfields and their interactions. Even though the deformation (1.1) breaks Lorentz invariance, $\int d^2\theta W(\Phi)$ turns out to be invariant. $\int d^2\overline{\theta}\overline{W}(\overline{\Phi})$ turns out to be independent of C.

In section 4 we study vector superfields. Because of the deformation (1.1), the standard gauge transformation rules are deformed. However, a simple change of variables allows us to use standard component fields with standard gauge transformation rules. In the Wess-Zumino gauge we find that the standard lagrangian $i\tau \int d^2\theta \,\mathrm{tr}\, WW - i\overline{\tau}\int d^2\overline{\theta}\,\mathrm{tr}\,\overline{W}\overline{W}$ is deformed by $(i\tau - i\overline{\tau})\left(-iC^{\mu\nu}\,\mathrm{tr}\,F_{\mu\nu}\overline{\lambda}\overline{\lambda} + \frac{|C|^2}{4}\,\mathrm{tr}(\overline{\lambda}\overline{\lambda})^2\right)$ + total derivative, where $C^{\mu\nu} \equiv C^{\alpha\beta}\epsilon_{\beta\gamma}\sigma_\alpha^{\mu\nu\,\gamma}$.

Section 5 is devoted to the study of the various rings. The chiral ring of the commutative theory is ruined by the deformation. However, the antichiral ring is still present even though we have only half of the supersymmetry. The antichiral ring retains all of the standard properties which it has in the commutative theory. In section 6 we study instantons and anti-instantons in these theories. Finally, in section 7 we show how these theories arise on branes in the presence of a background graviphoton.

2. Superspace

We want to consider the consequences of the nontrivial anticommutator

$$\{\theta^\alpha, \theta^\beta\} = C^{\alpha\beta} \tag{2.1}$$

on superspace. Because of (2.1), functions of θ should be ordered. We use Weyl ordering. This means that the function $\theta^\alpha\theta^\beta$ in the commutative space is ordered as $\frac{1}{2}[\theta^\alpha, \theta^\beta] = -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta$ (we follow the conventions of Wess and Bagger [1]) in the noncommutative space. When functions of θ are multiplied, the result should be reordered. As in ordinary noncommutative geometry, this is implemented by the star product

$$\begin{aligned}
f(\theta) * g(\theta) &= f(\theta)\exp\left(-\frac{C^{\alpha\beta}}{2}\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\partial}}{\partial\theta^\beta}\right)g(\theta) \\
&= f(\theta)\left(1 - \frac{C^{\alpha\beta}}{2}\frac{\overleftarrow{\partial}}{\partial\theta^\alpha}\frac{\overrightarrow{\partial}}{\partial\theta^\beta} - \det C\frac{\overleftarrow{\partial}}{\partial\theta\theta}\frac{\overrightarrow{\partial}}{\partial\theta\theta}\right)g(\theta) \\
\frac{\overrightarrow{\partial}}{\partial\theta^\beta}\theta^\alpha &\equiv \frac{\partial}{\partial\theta^\beta}\theta^\alpha = \delta^\alpha_\beta \\
\theta^\alpha\frac{\overleftarrow{\partial}}{\partial\theta^\beta} &\equiv -\delta^\alpha_\beta \\
\frac{\partial}{\partial\theta\theta} &\equiv \frac{1}{4}\epsilon^{\alpha\beta}\frac{\partial}{\partial\theta^\alpha}\frac{\partial}{\partial\theta^\beta}
\end{aligned} \tag{2.2}$$

If $f(\theta)$ is a bosonic function, $f(\theta)\frac{\overleftarrow{\partial}}{\partial\theta^\beta} = \frac{\overrightarrow{\partial}}{\partial\theta^\beta}f(\theta)$. Since $\frac{\overleftarrow{\partial}}{\partial\theta\theta}$ and $\frac{\overrightarrow{\partial}}{\partial\theta\theta}$ are bosonic, their definitions are obvious.

Useful examples are

$$\begin{aligned}
\theta^\alpha * \theta^\beta &= -\frac{1}{2}\epsilon^{\alpha\beta}\theta\theta + \frac{1}{2}C^{\alpha\beta} \\
\theta^\alpha * \theta\theta &= C^{\alpha\beta}\theta_\beta \\
\theta\theta * \theta^\alpha &= -C^{\alpha\beta}\theta_\beta \\
\theta\theta * \theta\theta &= -\frac{1}{2}\epsilon_{\alpha\beta}\epsilon_{\alpha'\beta'}C^{\alpha\alpha'}C^{\beta\beta'} = -\det C
\end{aligned} \tag{2.3}$$

If f or g depend on additional variables which have nontrivial commutation relations, the expression (2.2) should be modified appropriately.

Regarding the other coordinates, consider first simplest possibility that $\overline{\theta}^{\dot\alpha}$ satisfies standard commutation relations

$$\left\{\overline{\theta}^{\dot\alpha}, \overline{\theta}^{\dot\beta}\right\} = \left\{\overline{\theta}^{\dot\alpha}, \theta^\beta\right\} = \left[\overline{\theta}^{\dot\alpha}, x^\mu\right] = 0\,. \tag{2.4}$$

This means that $\overline{\theta}$ is not the complex conjugate of θ, which is possible only in euclidean space. We will be working on euclidean $\mathbb{R}^4$, but we will continue to use the lorentzian signature notation. Since $\overline{\theta}$ anticommutes with θ, we find the following useful identity

$$\theta\sigma^\mu\overline{\theta} * \theta\sigma^\nu\overline{\theta} = -\frac{1}{2}\eta^{\mu\nu}\theta\theta\overline{\theta}\overline{\theta} - \frac{1}{2}\overline{\theta}\overline{\theta}C^{\mu\nu} \tag{2.5}$$

where

$$C^{\mu\nu} \equiv C^{\alpha\beta}\epsilon_{\beta\gamma}\sigma^{\mu\nu\ \gamma}_{\alpha} \tag{2.6}$$

is selfdual. Useful identities are

$$\begin{aligned}
C^{\alpha\beta} &= \frac{1}{2}\epsilon^{\alpha\gamma}\sigma^{\mu\nu\ \beta}_{\gamma}C_{\mu\nu} \\
|C|^2 &\equiv C^{\mu\nu}C_{\mu\nu} = 4\det C\,.
\end{aligned} \tag{2.7}$$

What about the commutation relations of x? The simplest possibility is $[x^\mu, x^\nu] = [x^\mu, \theta^\alpha] = 0$. This makes it difficult to define chiral and antichiral fields because the ordinary D and $\overline{D}$ do not act as derivations (see below). Instead, using

$$y^\mu = x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha} \tag{2.8}$$

we can accompany (2.1)(2.4) with

$$[y^\mu, y^\nu] = [y^\mu, \theta^\alpha] = \left[y^\mu, \overline{\theta}^{\dot\alpha}\right] = 0\,. \tag{2.9}$$

This means that

$$\begin{aligned}
[x^\mu, \theta^\alpha] &= iC^{\alpha\beta}\sigma^\mu_{\beta\dot\alpha}\overline{\theta}^{\dot\alpha} \\
[x^\mu, x^\nu] &= \overline{\theta}\overline{\theta}C^{\mu\nu}
\end{aligned} \tag{2.10}$$

When functions on superspace are expressed in terms of $y, \theta, \overline{\theta}$ we can use the star product (2.2) where the derivatives with respect to θ are at fixed y and $\overline{\theta}$. We will follow this convention and will take $\partial/\partial\theta$ to mean a derivative at fixed y and $\overline{\theta}$.

We can now define the covariant derivatives. Since our θ and $\overline{\theta}$ derivatives are at fixed y (rather than at fixed x), the standard expressions

$$\begin{aligned} D_\alpha &= \frac{\partial}{\partial\theta^\alpha} + 2i\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\frac{\partial}{\partial y^\mu} \\ \overline{D}_{\dot\alpha} &= -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} \end{aligned} \tag{2.11}$$

can still be used. Clearly, they satisfy

$$\begin{aligned} \{D_\alpha, D_\beta\} &= 0 \\ \{\overline{D}_{\dot\alpha}, \overline{D}_{\dot\beta}\} &= 0 \\ \{\overline{D}_{\dot\alpha}, D_\alpha\} &= -2i\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial y^\mu} \end{aligned} \tag{2.12}$$

exactly as in the commutative space with $C = 0$.

The supercharges of the commutative theories are

$$\begin{aligned} Q_\alpha &= \frac{\partial}{\partial\theta^\alpha} \\ \overline{Q}_{\dot\alpha} &= -\frac{\partial}{\partial\overline{\theta}^{\dot\alpha}} + 2i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial y^\mu}\,, \end{aligned} \tag{2.13}$$

where again the derivatives with respect to θ and $\overline{\theta}$ are taken at fixed y. They satisfy

$$\begin{aligned} \{D_\alpha, Q_\beta\} &= \{\overline{D}_{\dot\alpha}, Q_\beta\} = \{D_\alpha, \overline{Q}_{\dot\beta}\} = \{\overline{D}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} = 0 \\ \{Q_\alpha, Q_\beta\} &= 0 \\ \{\overline{Q}_{\dot\alpha}, Q_\alpha\} &= 2i\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial y^\mu} \\ \{\overline{Q}_{\dot\alpha}, \overline{Q}_{\dot\beta}\} &= -4C^{\alpha\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\frac{\partial^2}{\partial y^\mu\partial y^\nu}\,. \end{aligned} \tag{2.14}$$

All these but the last one are as in the commutative space ($C = 0$). The star product (2.2) is invariant under Q and therefore we expect it to be a symmetry of the space. However, since $\overline{Q}$ depends explicitly on θ, it is clear that the star product is not invariant under $\overline{Q}$. Therefore, $\overline{Q}$ is not a symmetry of the noncommutative space.[1] Since half of $\mathcal{N} = 1$ supersymmetry is broken, we can refer to the unbroken Q supersymmetry as $\mathcal{N} = 1/2$ supersymmetry.

Reasonable conditions to impose on various generalizations of this superspace is that D and $\overline{D}$ of (2.11) act as derivations on functions of superspace and that they continue to satisfy (2.12). These conditions clearly forbid a deformation of the commutators of $\overline{\theta}$ (2.4), because $\overline{\theta}$ appears explicitly in D. However, there is still freedom in deforming (2.9) to

$$[y^\mu, y^\nu] = i\Theta^{\mu\nu}\,, \qquad [y^\mu, \theta^\alpha] = \Psi^{\mu\alpha}\,, \tag{2.15}$$

[1] We could have attempted to add to $\overline{Q}_{\dot\alpha}$ terms of the form $C^{\alpha\beta}\sigma^\mu_{\alpha\dot\alpha}\sigma^\nu_{\beta\dot\beta}\overline{\theta}^{\dot\beta}\frac{\partial^2}{\partial y^\mu\partial y^\nu}$ and $C^{\alpha\beta}\sigma^\mu_{\alpha\dot\alpha}\frac{\partial^2}{\partial y^\mu\partial\theta^\beta}$ to remove the last term in (2.14). Of particular interest is adding $-iC^{\alpha\beta}\sigma^\mu_{\alpha\dot\alpha}\frac{\partial^2}{\partial y^\mu\partial\theta^\beta}$ which makes all the commutation relations (2.14), as in the commutative space. The problem with such modifications of $\overline{Q}$ is that they include second derivatives, and therefore they do not act on products of fields as derivations.

with $\Theta^{\mu\nu}$ and $\Psi^{\mu\alpha}$ commuting and anticommuting c-numbers independent of y, θ and $\overline{\theta}$. $\Theta^{\mu\nu}$ has the effect of standard noncommutativity [2]. Here we will not explore these deformations and will set them to zero.

3. Chiral superfields

Chiral superfields are defined to satisfy $\overline{D}_{\dot\alpha}\Phi = 0$. This means that Φ is independent of $\overline{\theta}$. In components it is

$$\Phi(y,\theta) = A(y) + \sqrt{2}\,\theta\psi(y) + \theta\theta F(y)\,. \tag{3.1}$$

Since

$$\theta\theta = \theta^\alpha\theta_\alpha = -\theta^1\theta^2 + \theta^2\theta^1\,, \tag{3.2}$$

this expression is Weyl ordered. Two chiral superfields $\Phi_1(y,\theta)$ and $\Phi_2(y,\theta)$ are multiplied using the star product (2.2). Clearly, the result is a function of y and θ and therefore it is a chiral superfield

$$\begin{aligned}\Phi_1(y,\theta) * \Phi_2(y,\theta) \;=\; & \Phi_1(y,\theta)\Phi_2(y,\theta) - C^{\alpha\beta}\psi_{\alpha 1}(y)\psi_{\beta 2}(y) + \\ & +\sqrt{2}\,C^{\alpha\beta}\theta_\beta(\psi_{\alpha 1}(y)F_2(y) - \psi_{\alpha 2}(y)F_1(y)) - \\ & -\det C F_1(y)F_2(y)\,.\end{aligned} \tag{3.3}$$

Antichiral superfields are defined to satisfy $D_\alpha\overline{\Phi} = 0$. This means that $\overline{\Phi}$ depends only on $\overline{\theta}$ and

$$\overline{y}^\mu = y^\mu - 2i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}. \tag{3.4}$$

Since

$$[\overline{y}^\mu, \overline{y}^\nu] = 4\overline{\theta}\overline{\theta}C^{\mu\nu}\,, \tag{3.5}$$

antichiral superfields $\overline{\Phi}(\overline{y},\overline{\theta})$ have to be ordered. One possibility is to express them in terms of y and θ and to Weyl order the θs

$$\begin{aligned}\overline{\Phi}(y - 2i\theta\sigma\overline{\theta},\overline{\theta}) \;&=\; \overline{A}(y - 2i\theta\sigma\overline{\theta}) + \sqrt{2}\overline{\theta}\overline{\psi}(y - 2i\theta\sigma\overline{\theta}) + \overline{\theta}\overline{\theta}\overline{F}(y) \\ &=\; \overline{A}(y) + \sqrt{2}\overline{\theta}\overline{\psi}(y) - 2i\theta\sigma^\mu\overline{\theta}\partial_\mu\overline{A}(y) + \\ &\quad +\overline{\theta}\overline{\theta}\left(\overline{F}(y) + i\sqrt{2}\theta\sigma^\mu\partial_\mu\overline{\psi}(y) + \theta\theta\partial^2\overline{A}(y)\right).\end{aligned} \tag{3.6}$$

Alternatively, we can order the $\overline{y}$s and use the fact that the antichiral superfields do not have explicit θ dependence to multiply them as

$$\begin{aligned}\overline{\Phi}_1(\overline{y},\overline{\theta}) * \overline{\Phi}_2(\overline{y},\overline{\theta}) \;&=\; \overline{\Phi}_1(\overline{y},\overline{\theta}) \exp\left(2\overline{\theta}\overline{\theta}C^{\mu\nu}\frac{\overleftarrow{\partial}}{\partial\overline{y}^\mu}\frac{\overrightarrow{\partial}}{\partial\overline{y}^\nu}\right)\overline{\Phi}_2(\overline{y},\overline{\theta}) \\ &=\; \overline{\Phi}_1(\overline{y},\overline{\theta})\overline{\Phi}_2(\overline{y},\overline{\theta}) + 2\overline{\theta}\overline{\theta}C^{\mu\nu}\frac{\partial}{\partial\overline{y}^\mu}\overline{\Phi}_1(\overline{y},\overline{\theta})\frac{\partial}{\partial\overline{y}^\mu}\overline{\Phi}_2(\overline{y},\overline{\theta})\end{aligned} \tag{3.7}$$

where $\partial/\partial\overline{y}$ is taken at fixed $\overline{\theta}$, but unlike our conventions above, it is not at fixed y. Clearly, the result is an antichiral superfield.

Using these superfields and star product multiplication we can write a Wess-Zumino lagrangian. The simplest one is[2]

$$
\begin{aligned}
\mathcal{L} &= \int d^4\theta \; \overline{\Phi}\Phi + \int d^2\theta \left(\frac{1}{2}m\Phi * \Phi + \frac{1}{3}g\Phi * \Phi * \Phi\right) + \int d^2\overline{\theta} \left(\frac{1}{2}\overline{m}\overline{\Phi} * \overline{\Phi} + \frac{1}{3}\overline{g}\overline{\Phi} * \overline{\Phi} * \overline{\Phi}\right) \\
&= \mathcal{L}(C=0) - \frac{1}{3}g \det CF^3 + \text{total derivative}\,.
\end{aligned}
\tag{3.8}
$$

The only correction due to the noncommutativity arises from $\int d^2\theta \; \Phi * \Phi * \Phi$. It is amusing that even though our deformed superspace is not Lorentz invariant, the lagrangian (3.8) is Lorentz invariant.

Q and $\overline{Q}$ of (2.13) act on the fields (3.1)(3.6) exactly as in the commutative theory with $C = 0$. Therefore, if we want to examine the symmetries of the noncommutative theory which is based on the lagrangian (3.8), it is enough to examine the symmetries of the new term $\det CF^3$. Clearly, Q is still a symmetry, but $\overline{Q}$ is broken by the noncommutativity (nonzero C). This is in accord with the general expectation based on the symmetries of the star product.

More generally, there can be several chiral fields Φ_i and several antichiral fields $\overline{\Phi}_i$ which couple through $\int d^2\theta \, W(\Phi_i) + \int d^2\overline{\theta} \, \overline{W}(\overline{\Phi}_i)$. The functions W and $\overline{W}$ are polynomials (or can be locally holomorphic and antiholomorphic, respectively). Consider for example the monomials $W = \prod_i \Phi_{n_i}$, $\overline{W} = \prod_i \overline{\Phi}_{n_i}$. In the commutative theory the superfields commute and the order of multiplication is not important. This is no longer the case in the noncommutative theory: $\Phi_1 * \Phi_2 \neq \Phi_2 * \Phi_1$, $\overline{\Phi}_1 * \overline{\Phi}_2 \neq \overline{\Phi}_2 * \overline{\Phi}_1$. Therefore, there are several different noncommutative theories which correspond to the same commutative theory. They are parametrized by the coefficients of monomials which are not totally symmetrized. This is analogous to the standard operator ordering ambiguity in quantizing a classical theory. The same classical theory can lead to different quantum theories whose hamiltonians differ by terms of order $\hbar$.

It is natural to limit the discussion to noncommutative theories in which these new parameters vanish. Equivalently, we symmetrize (Weyl order) all the products of superfields in the superpotential.

Let us consider first $\int d^2\overline{\theta} \; \overline{W}(\overline{\Phi}_i)$. Without loss of generality we can focus on a monomial $\overline{W} = \prod^*_i \overline{\Phi}_{n_i}$ where $\prod^*$ stands for a symmetrized star product. It is clear from (3.7) that every power of C comes with $\overline{\theta\theta}$. Therefore, there can be at most one factor of C. It is also clear from (3.7) that the term with C is antisymmetric under the exchange of the two superfields, and therefore it vanishes in the symmetrized product. We conclude that $\int d^2\overline{\theta} \; \overline{W}(\overline{\Phi}_i)$ is not deformed in the noncommutative theory.

$\int d^2\theta \; W(\Phi_i)$ is a bit more interesting. Consider again a monomial $W = \prod^*_i \Phi_{n_i}$. Using (3.3) we can expand it in powers of C. Each Φ_n has at most two $\frac{\partial}{\partial\theta^\alpha}$ derivatives acting on it. Each factor with two derivatives $\frac{\partial^2}{\partial\theta^\alpha\partial\theta^\beta}\Phi_n = -2\epsilon_{\alpha\beta}\frac{\partial}{\partial\theta\theta}\Phi_n$ is a Lorentz singlet. The factors with one derivative are of the form $\frac{\partial}{\partial\theta^\alpha}\Phi_n$. Since the index n is symmetrized, the index α must be antisymmetrized. Therefore, there are at most two such factors and they are coupled to a Lorentz singlet. Since all the derivatives are coupled to a Lorentz

[2]We normalize $\int d^2\theta \; \theta\theta = \int d^2\overline{\theta} \; \overline{\theta\theta} = \int d^4\theta \; \theta\theta\overline{\theta\theta} = 1$.

singlet, the same should be true for the product of C's. We conclude that there is an even number of C's and they are coupled to a Lorentz singlet; i.e. the deformation depends only on $\det C$. In other words, even though C breaks Lorentz invariance, the theory with an arbitrary superpotential is Lorentz invariant.

4. Vector superfields

Vector superfields V describe gauge fields. In general V is a matrix and the gauge symmetry acts as[3]

$$e^V \to e^{V'} = e^{-i\overline{\Lambda}} e^V e^{i\Lambda}\,, \tag{4.1}$$

or infinitesimally

$$\delta e^V = -i\overline{\Lambda} e^V + i e^V \Lambda\,, \tag{4.2}$$

where Λ and $\overline{\Lambda}$ are matrices of chiral and antichiral superfields respectively. In our case we still use (4.1)(4.2) but the multiplication is the star product (e^V means a power series in V, where each term is multiplied using the star product).[4] It is important for the consistency of the theory that our star product is such that a function of (anti)chiral superfields is (anti)chiral. We can use the standard expressions for the chiral and antichiral field strength superfields

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\overline{D}\overline{D} e^{-V} D_\alpha e^V \\ \overline{W}_{\dot\alpha} &= \frac{1}{4} D D e^V \overline{D}_{\dot\alpha} e^{-V}\,, \end{aligned} \tag{4.3}$$

but with star products. As for $C = 0$ they transform under (4.1) as

$$\begin{aligned} W_\alpha &\to e^{-i\Lambda} W_\alpha e^{i\Lambda} \\ \overline{W}_{\dot\alpha} &\to e^{-i\overline{\Lambda}} \overline{W}_{\dot\alpha} e^{i\overline{\Lambda}}\,, \end{aligned} \tag{4.4}$$

or infinitesimally

$$\begin{aligned} \delta W_\alpha &= -i[\Lambda, W_\alpha] \\ \delta \overline{W}_{\dot\alpha} &= -i\left[\overline{\Lambda}, \overline{W}_{\dot\alpha}\right]. \end{aligned} \tag{4.5}$$

The gauge freedom can be used to set some of the components of V to zero. For $C = 0$ a convenient choice is the Wess-Zumino gauge. We generalize it to the noncommutative theory as

$$\begin{aligned} V(y,\theta,\overline{\theta}) &= -\theta\sigma^\mu\overline{\theta} A_\mu(y) + i\theta\theta\overline{\theta}\overline{\lambda}(y) - i\overline{\theta}\overline{\theta}\theta^\alpha\left(\lambda_\alpha(y) + \frac{1}{4}\epsilon_{\alpha\beta}C^{\beta\gamma}\sigma^\mu_{\gamma\dot\gamma}\{\overline{\lambda}^{\dot\gamma}, A_\mu\}\right) + \\ &\quad + \frac{1}{2}\theta\theta\overline{\theta}\overline{\theta}\left[D(y) - i\partial_\mu A^\mu(y)\right]. \end{aligned} \tag{4.6}$$

[3]As we said above, we follow the conventions of Wess and Bagger [1]. More standard conventions for gauge fields are obtained with the substitution $V \to -2V$.

[4]In order not to clutter the equations we will suppress the star symbol. It will be clear which product is a star product.

For nonzero C we parameterized the $\overline{\theta}\overline{\theta}\theta$ term differently than in the commutative theory. The reason for this will become clear momentarily.

Because of the noncommutativity the powers of V (4.6) are deformed from their standard expressions

$$\begin{aligned} V^2 &= \overline{\theta\theta}\left[-\frac{1}{2}\theta\theta A_\mu A^\mu - \frac{1}{2}C^{\mu\nu}A_\mu A_\nu + \frac{i}{2}\theta_\alpha C^{\alpha\beta}\sigma^\mu_{\beta\dot\alpha}\left[A_\mu, \overline{\lambda}^{\dot\alpha}\right] - \frac{1}{8}|C|^2\overline{\lambda\lambda}\right] \\ V^3 &= 0\,. \end{aligned} \tag{4.7}$$

The remaining infinitesimal gauge symmetry (4.2), which preserves the gauge choice (4.6) is

$$\begin{aligned} \Lambda(y,\theta) &= -\varphi(y) \\ \overline{\Lambda}(\overline{y},\theta) &= -\varphi(\overline{y}) - \frac{i}{2}\overline{\theta\theta}C^{\mu\nu}\{\partial_\mu\varphi, A_\nu\} \\ &= -\varphi(y) + 2i\theta\sigma^\mu\overline{\theta}\partial_\mu\varphi(y) - \theta\theta\overline{\theta\theta}\partial^2\varphi(y) - \frac{i}{2}\overline{\theta\theta}C^{\mu\nu}\{\partial_\mu\varphi, A_\nu\}\,. \end{aligned} \tag{4.8}$$

Note the new term proportional to C in $\overline{\Lambda}$ which is needed in order to preserve (4.6). It is straightforward to check that this transformation acts on the component fields as

$$\begin{aligned} \delta A_\mu &= -2\partial_\mu\varphi + i[\varphi, A_\mu] \\ \delta\lambda_\alpha &= i[\varphi, \lambda_\alpha] \\ \delta\overline{\lambda}_{\dot\alpha} &= i[\varphi, \overline{\lambda}_{\dot\alpha}] \\ \delta D &= i[\varphi, D] \end{aligned} \tag{4.9}$$

i.e. as standard gauge transformations.

This discussion of the gauge symmetry explains why we parameterized the coefficient of $\overline{\theta}\overline{\theta}\theta$ in V as we did. We wanted to ensure that the component fields transform as in ordinary gauge theory. This parametrization and the special Λ and $\overline{\Lambda}$ (4.8) are similar in spirit to the change of variables of [17] from noncommutative to commutative gauge fields in ordinary noncommutative gauge theories.

We now turn to the field strengths. Since in the gauge (4.6) V has at least one $\overline{\theta}$, the same is true for $D_\alpha V$. Using these facts, it is easy to show that $e^{-V}D_\alpha e^V = D_\alpha V + \frac{1}{2}[D_\alpha V, V]$ (all the products are star products), and therefore

$$\begin{aligned} W_\alpha &= -\frac{1}{4}\overline{D}^2 e^{-V}D_\alpha e^V = -\frac{1}{4}\overline{D}^2\left(D_\alpha V + \frac{1}{2}[D_\alpha V, V]\right) \\ &= -i\lambda_\alpha(y) + \left[\delta_\alpha^\beta D(y) - i\sigma_\alpha^{\mu\nu\ \beta}\left(F_{\mu\nu}(y) + \frac{i}{2}C_{\mu\nu}\overline{\lambda\lambda}(y)\right)\right]\theta_\beta + \theta\theta\sigma^\mu_{\alpha\dot\alpha}\mathcal{D}_\mu\overline{\lambda}^{\dot\alpha}(y) \\ &= W_\alpha(C=0) + \epsilon_{\alpha\gamma}C^{\gamma\beta}\theta_\beta\overline{\lambda\lambda}(y)\,. \end{aligned} \tag{4.10}$$

In this normalization the field strength and the covariant derivative are

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + \tfrac{i}{2}[A_\mu, A_\nu] \;\; \mathcal{D}_\mu\lambda_\alpha = \partial_\mu\lambda_\alpha + \frac{i}{2}[A_\mu, \lambda_\alpha]\,. \tag{4.11}$$

As we commented above, a more standard convention for gauge fields is obtained with the substitution $V \to -2V$.

As a check, we note that under the infinitesimal gauge transformation (4.8)

$$\delta W_\alpha(y,\theta) = i[\varphi(y), W_\alpha(y,\theta)]\,. \tag{4.12}$$

In order to compute $\overline{W}_{\dot\alpha}$ we use the fact that in the gauge (4.6) we have $e^V\overline{D}_{\dot\alpha}e^{-V} = -\overline{D}_{\dot\alpha}V + \frac{1}{2}[\overline{D}_{\dot\alpha}V, V] + \frac{1}{2}V\overline{D}_{\dot\alpha}VV$, where all the products are star products. The last term vanishes in the commutative theory, but it contributes for nonzero C. A somewhat lengthy calculation leads to

$$\begin{aligned}\overline{W}_{\dot\alpha} &= \frac{1}{4}DDe^V\overline{D}_{\dot\alpha}e^{-V} = -\frac{1}{4}DD\left(\overline{D}_{\dot\alpha}V - \frac{1}{2}[\overline{D}_{\dot\alpha}V, V] - \frac{1}{2}V\overline{D}_{\dot\alpha}VV\right) \\ &= \overline{W}_{\dot\alpha}(C=0) - \overline{\theta}\overline{\theta}\left[\frac{C^{\mu\nu}}{2}\{F_{\mu\nu}, \overline{\lambda}_{\dot\alpha}\} + C^{\mu\nu}\{A_\nu, \mathcal{D}_\mu\overline{\lambda}_{\dot\alpha} - \frac{i}{4}[A_\mu, \overline{\lambda}_{\dot\alpha}]\} + \frac{i}{16}|C|^2\{\overline{\lambda}\overline{\lambda}, \overline{\lambda}_{\dot\alpha}\}\right].\end{aligned} \tag{4.13}$$

As a check, we note that the infinitesimal gauge transformation $\delta\overline{W}_{\dot\alpha} = -i[\overline{\Lambda}, \overline{W}_{\dot\alpha}]$ with a star product with $\overline{\Lambda}$ of (4.8) is

$$\delta\overline{W}_{\dot\alpha} = i[\varphi(y), \overline{W}_{\dot\alpha}] + \overline{\theta}\overline{\theta}C^{\mu\nu}\left(2\{\partial_\nu\varphi, \partial_\mu\overline{\lambda}_{\dot\alpha}\} + \frac{i}{2}\left[\{\partial_\nu\varphi, A_\mu\}, \overline{\lambda}_{\dot\alpha}\right]\right). \tag{4.14}$$

It is implemented by the standard gauge transformations on the component fields (4.9).

In order to find the lagrangian we need

$$\begin{aligned}\int d^2\theta\,\mathrm{tr}\,WW &= \int d^2\theta\ \mathrm{tr}\,WW(C=0) - iC^{\mu\nu}\,\mathrm{tr}\,F_{\mu\nu}\overline{\lambda}\overline{\lambda} + \frac{|C|^2}{4}\,\mathrm{tr}(\overline{\lambda}\overline{\lambda})^2 \\ \int d^2\overline{\theta}\,\mathrm{tr}\,\overline{W}\overline{W} &= \int d^2\overline{\theta}\,\mathrm{tr}\,\overline{W}\overline{W}(C=0) - iC^{\mu\nu}\,\mathrm{tr}\,F_{\mu\nu}\overline{\lambda}\overline{\lambda} + \frac{|C|^2}{4}\,\mathrm{tr}(\overline{\lambda}\overline{\lambda})^2 + \\ &\quad + \text{total derivative}\,.\end{aligned} \tag{4.15}$$

The deformations of $\int d^2\theta\,\mathrm{tr}\,WW$ and of $\int d^2\overline{\theta}\,\mathrm{tr}\,\overline{W}\overline{W}$ are equal up to total derivatives. Therefore, the deformation of the lagrangian depends only on $\mathrm{Im}\,\tau = 4\pi/g^2$. More precisely, the term multiplying $\mathrm{Re}\,\tau = \theta/2\pi$ remains a total derivative.

Since the unbroken supersymmetry acts as a simple differential operator $Q_\alpha = \partial/\partial\theta^\alpha$ (with fixed y), it is easy to see how the various fields transform

$$\begin{aligned}\delta\lambda &= i\epsilon D + \sigma^{\mu\nu}\epsilon\left(F_{\mu\nu} + \frac{i}{2}C_{\mu\nu}\overline{\lambda}\overline{\lambda}\right) \\ \delta A_\mu &= -i\overline{\lambda}\overline{\sigma}_\mu\epsilon \\ \delta F_{\mu\nu} &= i\epsilon\left(\sigma_\nu\mathcal{D}_\mu - \sigma_\mu\mathcal{D}_\nu\right)\overline{\lambda} \\ \delta D &= -\epsilon\sigma^\mu\mathcal{D}_\mu\overline{\lambda} \\ \delta\overline{\lambda} &= 0\,.\end{aligned} \tag{4.16}$$

The only effect of the deformation is the correction to $\delta\lambda$.

5. (Anti)chiral rings

In this section we will explore some of the consequences of the unbroken supersymmetry Q. We will examine whether we can define chiral and antichiral rings, and which of their properties in the commutative theory survive in the noncommutative theory.

We start by considering the antichiral operators (3.6) $\overline{\Phi}(\overline{y} = y - 2i\theta\sigma\overline{\theta}, \overline{\theta})$. Their correlation functions are

$$\left\langle \overline{\Phi}_1(\overline{y}_1, \overline{\theta}_1)\overline{\Phi}_2(\overline{y}_2, \overline{\theta}_2) \ldots \overline{\Phi}_n(\overline{y}_n, \overline{\theta}_n) \right\rangle = \overline{F}\left(\overline{y}_i = y_i - 2i\theta_i\sigma\overline{\theta}_i, \overline{\theta}_i\right). \tag{5.1}$$

Translation invariance shows that $\overline{F}$ depends only on $\overline{y}_i - \overline{y}_j$. If the vacuum is Q invariant, we derive the Ward identity

$$\sum_i \frac{\partial}{\partial\theta_i}\overline{F} = -2i\sigma^\mu \sum_i \overline{\theta}_i \frac{\partial}{\partial \overline{y}_i^\mu}|_{\overline{\theta}_i}\overline{F} = 0\,. \tag{5.2}$$

This means that $\overline{F}(\overline{y}_i, \overline{\theta}_i = 0)$ is a constant independent of $\overline{y}_i$. Therefore, the correlation functions of first components of anti-chiral operators are independent of separations. Following the standard argument based on cluster decomposition, the correlation function (5.1) factorizes

$$\left\langle \overline{\Phi}_1(\overline{y}_1, \overline{\theta}_1 = 0)\overline{\Phi}_2(\overline{y}_2, \overline{\theta}_2 = 0) \ldots \overline{\Phi}_n(\overline{y}_n, \overline{\theta}_n = 0) \right\rangle = \prod_{i=1}^n \langle \overline{\Phi}_i(\overline{y}_i, \overline{\theta}_i = 0)\rangle\,, \tag{5.3}$$

where each expectation value in the right hand side is independent of $\overline{y}_i$. These correlation functions do not exhibit the full complexity of quantum field theory involving short distance singularities etc. In other words, it is easy to multiply such operators and they form a ring.

The set of antichiral operators which are D derivatives of other operators can be set to zero in the ring. To see that, consider the correlation function of the antichiral operator $\overline{\Phi}_1 = D_\alpha \Psi(\overline{y}_1, \theta_1, \overline{\theta}_1)$ for some α and Ψ. In this case the correlation function (5.1) can be written as

$$\begin{aligned}
\overline{F}\left(\overline{y}_1, \ldots, \overline{y}_n, \overline{\theta}_1, \ldots, \overline{\theta}_n\right) &= \left(\frac{\partial}{\partial\theta_1^\alpha} + 2i\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}_1^{\dot\alpha}\frac{\partial}{\partial y_1^\mu}\right) G(\theta_1, \overline{y}_1, \ldots, \overline{y}_n, \overline{\theta}_1, \ldots, \overline{\theta}_n) \\
&= 2i\sigma^\mu_{\alpha\dot\alpha}\sum_{i=1}^n \overline{\theta}_i^{\dot\alpha}\frac{\partial}{\partial\overline{y}_i^\mu}|_{\theta_i, \overline{\theta}_i} G\left(\theta_1, \overline{y}_1, \ldots, \overline{y}_n, \overline{\theta}_1, \ldots, \overline{\theta}_n\right),
\end{aligned} \tag{5.4}$$

where $G = \langle \Psi(\overline{y}_1, \theta_1, \overline{\theta}_1)\overline{\Phi}_2(\overline{y}_2, \overline{\theta}_2) \ldots \overline{\Phi}_n(\overline{y}_n, \overline{\theta}_n)\rangle$ can have explicit θ_1 dependence. In the last step in (5.4) we used the $\mathcal{N} = \frac{1}{2}$ supersymmetry (Q supersymmetry) Ward identity (5.2). It follows from (5.4) that for such $\overline{\Phi}_1$ the correlation function $\overline{F}(\overline{y}_1, \ldots, \overline{y}_n, \overline{\theta}_1 = 0, \ldots, \overline{\theta}_n = 0) = 0$, and therefore we can mod out the set of operators $\{\overline{\Phi}_i(\overline{y}, \overline{\theta} = 0)\}$ by those operators which are of the form $\overline{\Phi} = D_\alpha\Psi$.

The antichiral operators form a ring with such an ideal in the commutative theory. Surprisingly, this is also true in the noncommutative theory. This is closely related to our discussion above explaining that the contribution of the anti-superpotential $\int d^2\overline{\theta}\overline{W}$ to the action is not deformed by C. However, since $\int d^2\overline{\theta}\,\mathrm{tr}\,\overline{W}_{\dot\alpha}\overline{W}^{\dot\alpha}$ is deformed by C, the ring is not exactly the same as in the commutative theory.

It is instructive to repeat this discussion from a somewhat different point of view. Instead of defining antichiral operators as we did, we can define antichiral operators $\overline{\mathcal{O}}$ by $[Q, \overline{\mathcal{O}}\} = 0$. Since Q is conjugate to D this definition is conjugate to the one we used above. Now it is easy to prove, using the fact that Q is a symmetry of the theory, that if the vacuum is invariant under Q, the correlation function

$$\left\langle [Q,M\}\overline{\mathcal{O}}_2\ldots\overline{\mathcal{O}}_n\right\rangle = \pm\left\langle \overline{\mathcal{O}}_1[Q,\overline{\mathcal{O}}_2\}\ldots\overline{\mathcal{O}}_n\right\rangle \pm\ldots\pm\left\langle \overline{\mathcal{O}}_1\overline{\mathcal{O}}_2\ldots[Q,\overline{\mathcal{O}}_n\}\right\rangle = 0 \qquad (5.5)$$

(the signs $\pm$ depend on whether M and $\mathcal{O}_i$ are bosonic or fermionic), and therefore the antichiral operators of the form $[Q,M\}$ can be set to zero in the correlation functions. Therefore, we can identify $\mathcal{O}\sim\mathcal{O}+[Q,M\}$ for every M. Furthermore, even though $\overline{Q}$ of (2.13) is not a symmetry, since (2.14) $\{\overline{Q}_{\dot\alpha},Q_\alpha\} = 2i\sigma^\mu_{\alpha\dot\alpha}\frac{\partial}{\partial y^\mu}$, it is still true that for an antichiral operator $\partial\overline{\mathcal{O}}\sim[Q,[\overline{Q},\overline{\mathcal{O}}\}\}$. Using (5.5), the correlation functions of $\partial\overline{\mathcal{O}}$ with other antichiral operators vanish. Therefore, the correlation functions of antichiral operators are independent of separations and they factorize. We again conclude that the antichiral operators form a ring.

These facts are not true for the chiral operators. Consider the expectation value of a product of chiral operators

$$\langle\Phi_1(y_1,\theta_1)\Phi_2(y_2,\theta_2)\ldots\Phi_n(y_n,\theta_n)\rangle = F(y_1,\ldots,y_n,\theta_1,\ldots,\theta_n)\,. \qquad (5.6)$$

Translation invariance guarantees that F depends only on y_i-y_j. If the vacuum preserves our unbroken supersymmetry Q, the correlation function F is annihilated by $Q=\sum_i\frac{\partial}{\partial\theta_i}$, and therefore is a function of $\theta_i-\theta_j$. In the commutative theory we can also use invariance under $\overline{Q}=\sum_i\left(-\frac{\partial}{\partial\overline{\theta}_i}+2i\theta_i\sigma^\mu\frac{\partial}{\partial y_i^\mu}\right)$ to show that $F(y_1,\ldots,y_n,\theta_1=0,\ldots,\theta_n=0)$ is independent of y_i. But since $\overline{Q}$ is not a symmetry of the noncommutative theory we cannot argue that in our theory. Therefore, the correlation functions of first component of chiral superfields depend on separations, and these operators no longer form a ring. Similarly, since $\overline{Q}$ is not a symmetry, we cannot identify chiral operators which differ by $\overline{D}$ of another operator.

6. Instantons

In this section we study instantons and anti-instantons and their fermion zero modes. The deformation of the action (4.15) does not affect the purely bosonic terms. Therefore, the instanton equation $F^+=0$ with nonzero F^-, and the anti-instanton equation $F^-=0$ with nonzero F^+ are not modified.

In the commutative theory the instantons or the anti-instantons break half of the sueprsymmetry. The instantons with $F^+=0$ break the $\overline{Q}$ supersymmetries and the anti-instantons with $F^-=0$ break the Q supersymmetries. These broken charges lead to some fermion zero modes. In addition to these, there are other fermion zero modes whose existence follows from the index theorem.

Our deformation of superspace breaks supersymmetry to $\mathcal{N}=1/2$ supersymmetry — only Q is a symmetry but $\overline{Q}$ is broken. Unlike the breaking in an instanton background, which is spontaneous breaking, this breaking is explicit. Let us examine the interplay between these two effects. The anti-instantons spontaneously break the supersymmetry which is preserved by the deformation Q. Therefore, Q should lead to fermion zero modes.

The instantons, on the other hand, break a symmetry which is already broken by the deformation $\overline{Q}$. Therefore, this symmetry does not lead to zero modes. The unbroken supersymmetry Q pairs fermion and boson nonzero modes.

Let us see how these general considerations are compatible with the index theorem. The fermion equations of motion derived from the action (4.15) are

$$\begin{aligned}\sigma^{\mu}\mathcal{D}_{\mu}\overline{\lambda} &= 0\\ \overline{\sigma}^{\mu}\mathcal{D}_{\mu}\lambda &= -C^{\mu\nu}F^{+}_{\mu\nu}\overline{\lambda} - i\frac{|C|^2}{2}(\overline{\lambda}\overline{\lambda})\overline{\lambda}\,.\end{aligned} \tag{6.1}$$

An anti-instanton ($F^{-}=0$) has λ zero modes, and $\overline{\lambda}$ vanishes. Therefore the right hand side of the second equation is zero and the equations are exactly as in the commutative theory. The situation with instantons ($F^{+}=0$) is different. For zero C there are $\overline{\lambda}$ zero modes but no λ zero modes. When C is nonzero this remains true in accordance with the index theorem. However, at order $\overline{\lambda}^3$ we find a source in the λ equation and λ cannot remain zero. Therefore, the zero modes are not lifted but it appears to be difficult to introduce collective coordinates for them. This is consistent with the fact that they are not associated with a broken symmetry.

7. Relation to graviphoton background

In this section we show how our noncommutative superspace arises in string theory in background with constant graviphoton field strength. Our discussion is motivated by the work of Ooguri and Vafa [12], and is similar to it. But we differ from [12] in some crucial details.

We start with the holomorphic part of the heterotic strings using Berkovits' formalism (for a nice review see [18]). The relevant part of the worldsheet lagrangian is

$$\mathcal{L}_h = \frac{1}{\alpha'}\left(\frac{1}{2}\widetilde{\partial}x^{\mu}\partial x_{\mu} + p_{\alpha}\widetilde{\partial}\theta^{\alpha} + \overline{p}_{\dot\alpha}\widetilde{\partial}\overline{\theta}^{\dot\alpha}\right), \tag{7.1}$$

where we ignore the worldsheet fields ρ and ζ. Since we use a bar to denote the space time chirality, we use a tilde to denote the worldsheet chirality. Therefore when the worldsheet has euclidean signature it is parametrized by z and $\widetilde{z}$ which are complex conjugate of each other. For a lorentzian signature target space p_{α} is the hermitian conjugate of $-\overline{p}_{\dot\alpha}$. For a euclidean target space they are independent fields. p and $\overline{p}$ are canonically conjugate to θ and $\overline{\theta}$; they are the worldsheet versions of $-\frac{\partial}{\partial\theta}|_x$ and $-\frac{\partial}{\partial\overline{\theta}}|_x$. The reason x is held fixed in these derivatives is that x appears as another independent field in (7.1). We also define d_{α}, $\overline{d}_{\dot\alpha}$, q_{α} and $\overline{q}_{\dot\alpha}$, which are the worldsheet versions of D_{α}, $\overline{D}_{\dot\alpha}$, Q_{α} and $\overline{Q}_{\dot\alpha}$

$$\begin{aligned}d_{\alpha} &= -p_{\alpha} + i\sigma^{\mu}_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial x_{\mu} - \overline{\theta}\overline{\theta}\partial\theta_{\alpha} + \frac{1}{2}\theta_{\alpha}\partial(\overline{\theta}\overline{\theta})\\ \overline{d}_{\dot\alpha} &= \overline{p}_{\dot\alpha} - i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot\alpha}\partial x_{\mu} - \theta\theta\partial\overline{\theta}_{\dot\alpha} + \frac{1}{2}\overline{\theta}_{\dot\alpha}\partial(\theta\theta)\\ q_{\alpha} &= -p_{\alpha} - i\sigma^{\mu}_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial x_{\mu} + \frac{1}{2}\overline{\theta}\overline{\theta}\partial\theta_{\alpha} - \frac{3}{2}\partial(\theta_{\alpha}\overline{\theta}\overline{\theta})\\ \overline{q}_{\dot\alpha} &= \overline{p}_{\dot\alpha} + i\theta^{\alpha}\sigma^{\mu}_{\alpha\dot\alpha}\partial x_{\mu} + \frac{1}{2}\theta\theta\partial\overline{\theta}_{\dot\alpha} - \frac{3}{2}\partial(\overline{\theta}_{\dot\alpha}\theta\theta)\,.\end{aligned} \tag{7.2}$$

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Our definitions of q and $\overline{q}$ differ from the inetgrand of the supercharges in [18] by total derivatives which do not affect the charges, but are important for our purpose.

Since we are interested in working in terms of $y^\mu = x^\mu + i\theta^\alpha \sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}$, and the derivatives with fixed y, $Q_\alpha = \frac{\partial}{\partial\theta}|_y$ and $\overline{D}_{\dot\alpha} = -\frac{\partial}{\partial\overline{\theta}}|_y$, we change variables to y, q_α and $\overline{d}_{\dot\alpha}$. A simple calculation leads to

$$\mathcal{L}_h = \frac{1}{\alpha'}\left(\frac{1}{2}\widetilde{\partial} y^\mu \partial y_\mu - q_\alpha \widetilde{\partial}\theta^\alpha + \overline{d}_{\dot\alpha}\widetilde{\partial}\overline{\theta}^{\dot\alpha} + \text{total derivative}\right). \tag{7.3}$$

In deriving this expression the total derivatives in q in (7.2) is important.

In the type II theory (7.1) is replaced with

$$\mathcal{L}_{II} = \frac{1}{\alpha'}\left(\frac{1}{2}\widetilde{\partial} x^\mu \partial x_\mu + p_\alpha \widetilde{\partial}\theta^\alpha + \overline{p}_{\dot\alpha}\widetilde{\partial}\overline{\theta}^{\dot\alpha} + \widetilde{p}_\alpha \partial\widetilde{\theta}^\alpha + \widetilde{\overline{p}}_{\dot\alpha}\partial\widetilde{\overline{\theta}}^{\dot\alpha}\right), \tag{7.4}$$

where again the tilde denotes the worldsheet chirality. Changing variables to

$$\begin{aligned}
y^\mu &= x^\mu + i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha} + i\widetilde{\theta}^\alpha\sigma^\mu_{\alpha\dot\alpha}\widetilde{\overline{\theta}}^{\dot\alpha} \\
\overline{d}_{\dot\alpha} &= \overline{p}_{\dot\alpha} - i\theta^\alpha\sigma^\mu_{\alpha\dot\alpha}\partial x_\mu - \theta\theta\partial\overline{\theta}_{\dot\alpha} + \frac{1}{2}\overline{\theta}_{\dot\alpha}\partial(\theta\theta) \\
q_\alpha &= -p_\alpha - i\sigma^\mu_{\alpha\dot\alpha}\overline{\theta}^{\dot\alpha}\partial x_\mu + \frac{1}{2}\overline{\theta\theta}\partial\theta_\alpha - \frac{3}{2}\partial(\theta_\alpha\overline{\theta\theta}) \\
\widetilde{\overline{d}}_{\dot\alpha} &= \widetilde{\overline{p}}_{\dot\alpha} - i\widetilde{\theta}^\alpha\sigma^\mu_{\alpha\dot\alpha}\widetilde{\partial} x_\mu - \widetilde{\theta\theta}\widetilde{\partial}\widetilde{\overline{\theta}}_{\dot\alpha} + \frac{1}{2}\widetilde{\overline{\theta}}_{\dot\alpha}\widetilde{\partial}(\widetilde{\theta\theta}) \\
\widetilde{q}_\alpha &= -\widetilde{p}_\alpha - i\sigma^\mu_{\alpha\dot\alpha}\widetilde{\overline{\theta}}^{\dot\alpha}\widetilde{\partial} x_\mu + \frac{1}{2}\widetilde{\overline{\theta\theta}}\widetilde{\partial}\widetilde{\theta}_\alpha - \frac{3}{2}\widetilde{\partial}\left(\widetilde{\theta}_\alpha\widetilde{\overline{\theta\theta}}\right)
\end{aligned} \tag{7.5}$$

we derive

$$\mathcal{L}_{II} = \frac{1}{\alpha'}\left(\frac{1}{2}\widetilde{\partial} y^\mu \partial y_\mu - q_\alpha \widetilde{\partial}\theta^\alpha + \overline{d}_{\dot\alpha}\widetilde{\partial}\overline{\theta}^{\dot\alpha} - \widetilde{q}_\alpha \partial\widetilde{\theta}^\alpha + \widetilde{\overline{d}}_{\dot\alpha}\partial\widetilde{\overline{\theta}}^{\dot\alpha} + \text{total derivative}\right). \tag{7.6}$$

If the worldsheet ends on a D-brane, the boundary conditions are easily found by imposing that there is no surface term in the equations of motion. For a boundary at $z = \widetilde{z}$, we can use the boundary conditions $\theta(z = \widetilde{z}) = \widetilde{\theta}(z = \widetilde{z})$, $q(z = \widetilde{z}) = \widetilde{q}(z = \widetilde{z})$, etc. Then the solutions of the equations of motion are such that $\theta(z) = \widetilde{\theta}(\widetilde{z})$, $q(z) = \widetilde{q}(\widetilde{z})$, etc.; i.e. the fields extend to holomorphic fields beyond the boundary. The boundary breaks half the supersymmetries preserving only $\oint q dz + \oint \widetilde{q} d\widetilde{z}$ and $\oint \overline{q} dz + \oint \widetilde{\overline{q}} d\widetilde{z}$.

Motivated by [12] we consider the system in a background of constant graviphoton field strength $F_{\alpha\beta}$. We take the field strength F to be selfdual ($F_{\dot\alpha\dot\beta} = 0$) because such a background is a solution of the spacetime equations of motion without back reaction of the metric. This background is represented in the worldsheet lagrangian by adding to (7.4) or (7.6) the term

$$F^{\alpha\beta} q_\alpha \widetilde{q}_\beta\,. \tag{7.7}$$

The form (7.6) is particularly convenient because it makes it manifest that the worldsheet theory remains free in this background. It is important that the coordinates y are free and independent of the background, while the original standard spacetime coordinates x couple to F.

Ignoring the trivial fields we are led to consider the lagrangian

$$\frac{1}{\alpha'}\left(-q_\alpha \widetilde{\partial}\theta^\alpha - \widetilde{q}_\alpha \partial\widetilde{\theta}^\alpha + \alpha' F^{\alpha\beta} q_\alpha \widetilde{q}_\beta\right). \tag{7.8}$$

The fields q and $\widetilde{q}$ can easily be integrated out using their equations of motion

$$\begin{aligned} \widetilde{\partial}\theta^\alpha &= \alpha' F^{\alpha\beta}\widetilde{q}_\beta \\ \partial\widetilde{\theta}^\alpha &= -\alpha' F^{\alpha\beta} q_\beta \end{aligned} \tag{7.9}$$

leading to

$$\mathcal{L}_{\text{eff}} = \left(\frac{1}{\alpha'^2 F}\right)_{\alpha\beta} \partial\widetilde{\theta}^\alpha \widetilde{\partial}\theta^\beta . \tag{7.10}$$

When we consider a system with a boundary (along $z = \widetilde{z}$ for a euclidean worldsheet) we need to find the appropriate boundary conditions. These are determined from the surface term in the equations of motion

$$\left(\frac{1}{F}\right)_{\alpha\beta} \left(\widetilde{\partial}\theta^\alpha \delta\widetilde{\theta}^\beta + \partial\widetilde{\theta}^\alpha \delta\theta^\beta\right)|_{z=\widetilde{z}} = 0 . \tag{7.11}$$

We impose

$$\theta^a(z=\widetilde{z}) = \widetilde{\theta}^a(z=\widetilde{z}), \qquad \partial\widetilde{\theta}^a(z=\widetilde{z}) = -\widetilde{\partial}\theta^a(z=\widetilde{z}) . \tag{7.12}$$

The first condition states that the superspace has half the number of θs. The second condition guarantees, using (7.9) that $q_\alpha(z=\widetilde{z}) = \widetilde{q}_\alpha(z=\widetilde{z})$.

The various propagators are found by imposing the proper singularity at coincident points, Fermi statistics and the boundary conditions (7.12). We find

$$\begin{aligned} \langle\theta^\alpha(z,\widetilde{z})\theta^\beta(w,\widetilde{w})\rangle &= \frac{\alpha'^2 F^{\alpha\beta}}{2\pi i}\log\frac{\widetilde{z}-w}{z-\widetilde{w}} \\ \langle\widetilde{\theta}^\alpha(z,\widetilde{z})\widetilde{\theta}^\beta(w,\widetilde{w})\rangle &= \frac{\alpha'^2 F^{\alpha\beta}}{2\pi i}\log\frac{\widetilde{z}-w}{z-\widetilde{w}} \\ \langle\theta^\alpha(z,\widetilde{z})\widetilde{\theta}^\beta(w,\widetilde{w})\rangle &= \frac{\alpha'^2 F^{\alpha\beta}}{2\pi i}\log\frac{(z-w)(\widetilde{z}-\widetilde{w})}{(z-\widetilde{w})^2} . \end{aligned} \tag{7.13}$$

The branch cuts of the logarithms is outside the worldsheet. Therefore for two points on the boundary $z = \widetilde{z} = \tau$ and $w = \widetilde{w} = \tau'$

$$\langle\theta^\alpha(\tau)\theta^\beta(\tau')\rangle = \frac{\alpha'^2 F^{\alpha\beta}}{2}\text{sign}(\tau-\tau') . \tag{7.14}$$

Using standard arguments about open string coupling, this leads to

$$\{\theta^\alpha, \theta^\beta\} = \alpha'^2 F^{\alpha\beta} = C^{\alpha\beta} \neq 0 ; \tag{7.15}$$

i.e. to a deformation of the anticommutator of the θs. It is important that since the coordinates $\overline{\theta}$ and y were not affected by the background coupling (7.7), they remain commuting. In particular we derive that $[y^\mu, y^\nu] = 0$ and therefore $[x^\mu, x^\nu] \neq 0$, exactly as motivated earlier by consistency.

We conclude that the graviphoton background leads to exactly the same deformation of superspace which we considered above.

Since the equation of motion of θ states that q is holomorphic, as in the case without the background, it extends to a holomorphic field $\widetilde{q}(\widetilde{z}) = q(z)$, and therefore $\oint q dz + \oint \widetilde{q} d\widetilde{z}$ is a conserved charge. However, even though $\partial\widetilde{\theta} \sim q$ is holomorphic and $\widetilde{\partial}\theta \sim \widetilde{q}$ is antiholomorphic, θ and $\widetilde{\theta}$ are no longer holomorphic and do not extend holomorphically through the boundary (see the propagators (7.13)). Therefore, the supersymmetries $\oint \overline{q} dz + \oint \overline{\widetilde{q}} d\widetilde{z}$ are broken by the deformation.

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